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Spurious Numerical Solutions in Higher Dimensional Discrete Systems

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Introduction

IN the present paper, we discuss the qualitative features of spurious asymptotes which are found as numerical solutions on discretizing the original continuous partial differential equation (PDE). In the recent literature,[1,2] it is reported that chaotic solutions, even when the true solutions of the original differential equation approach limit cycles or fixed points, are often obtained as a consequence of the omission of the local discretization errors in the approximation of continuous differential equations by their discretized counterparts. The original and systematic work of Refs. 4–6 suggests that investigating typical features of such stable and unstable spurious numerical solutions (periodic points, limit cycles, tori, and chaotic motions), alias ghost solutions, not only could contribute to interpretations of numerical results in fluid dynamic study but may also provide useful knowledge about convergence speed for such steady-state solutions. This study presents analysis concerning behavior of spurious numerical solutions on applying a nonlinear dynamics approach. This supports the original work of Refs. 3–6. Furthermore, we discuss the dependence of difference schemes on nonlinear instability in cases of higher dimensional dynamical systems. The one-dimensional Burgers equation was selected for the analysis of partial differential equations representative of fluid dynamics.

Analysis

We consider the initial boundary value problem for the one-dimensional Burgers equation and its finite difference approximation. In this problem, in which boundary values are zero (fixed), only one trivial solution $[u(x) = 0]$ is allowed with the physical meaning.[7] Central spatial difference and Euler forward temporal difference schemes are applied

$$\begin{cases} u_i^{m+1} = u_i^m + \Delta t\left\{-u_i^m \dfrac{u_{i+1}^m - u_{i-1}^m}{2\Delta x} + \nu \dfrac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta x)^2}\right\} \\ \qquad [0 < i < (n-1), m \geq 0] \\ u_i^0 : \text{ initial data} \\ u_0^m = u_{n-1}^m = 0 \end{cases} \tag{1}$$

Figure 1 shows a comparison of bifurcation diagrams that result on using the spatial central difference scheme (1), where $\nu = \frac{1}{4}$ and $\Delta x = \frac{1}{8}$ are fixed. Initial data are $u_0^0 = 0.0, u_1^0 = 0.5, u_2^0 = 2.0, u_{n-2}^0 = -1.45, u_{n-1}^0 = 0.0$, and $u_i^0 = 2.0 - 3.45 \times (i-2)/(n-4)$

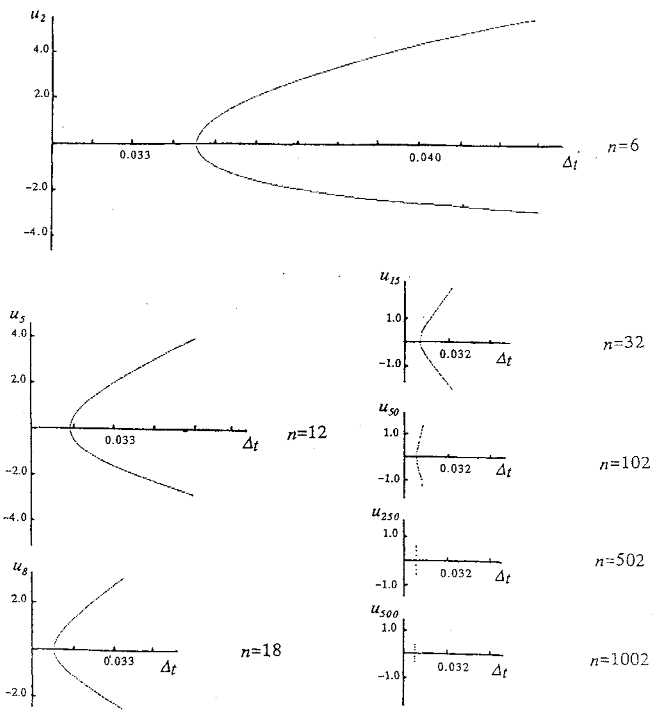

Fig. 1 Bifurcation diagrams of the $(n-2)$-points dynamical system using Eqs. (1) and (2a) discretization.

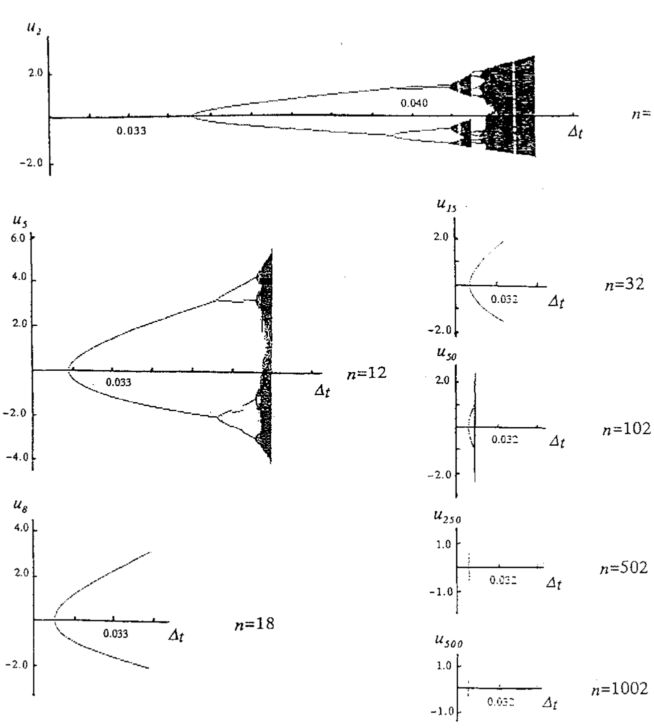

Fig. 2 Bifurcation diagrams for the $(n-2)$-points dynamical system using Eqs. (1) and (2b) discretization.

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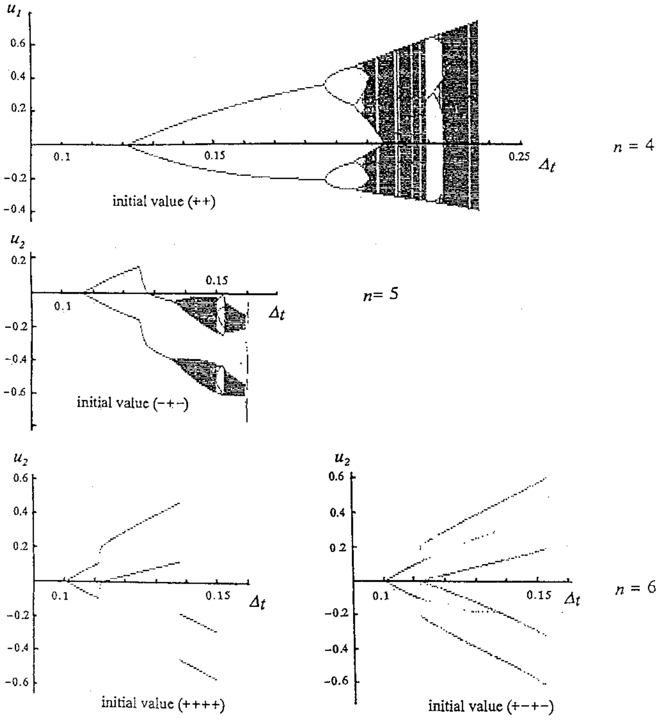

Fig. 3 Bifurcation diagrams for the (n−2)-points dynamical system using downwind scheme, Eq. (3) discretization.

($i = 3, 4, \ldots, n-3$). The midpoint value ($u_{n/2-1}$) is the ordinate. In all cases, only the two-periodic points appear as the spurious numerical solution (limit cycles can be seen in the case of $n = 5$) and the structure of the solutions remains simple as the domain increases. The Δt value where the fixed point bifurcates to the two-periodic points, Δt_{b1}, is given by the eigenvalue analysis of its Jacobian matrix and may be shown to be $\Delta t_{b1} \to [(\Delta x)^2/2\nu]$ as $n \to \infty$, i.e., the value corresponding to the stability limit for the linear diffusion equation. The Δt value of computational instability, a rapid and unbounded amplification of the variables, was also to approach $[(\Delta x)^2/2\nu]$ as n increases. Next, we discuss the dependence of the spatial discretization at the boundaries on the structure of the steady-state solutions. Equations (2a) and (2b) show possible spatial discretizations of the nonlinear and linear terms at the left and right boundaries. Equations (2a) are used in calculations for Fig. 1, and the fixed boundary values $u_0^m = u_{n-1}^m = 0.0$ are included in both of the nonlinear and linear terms. Conversely, Eqs. (2b) do not include the fixed boundary values in the nonlinear term, but include them only in the viscous term and were the basis for Fig. 2 using the same conditions as those in Fig. 1.

$$\begin{cases} u\dfrac{\partial u}{\partial x}\Big|_{i=1} = u_1^m \dfrac{u_2^m - 0}{2\Delta x} \\ \dfrac{\partial^2 u}{\partial x^2}\Big|_{i=1} = \dfrac{u_2^m - 2u_1^m + 0}{(\Delta x)^2} \\ u\dfrac{\partial u}{\partial x}\Big|_{i=n-2} = u_{n-2}^m \dfrac{0 - u_{n-3}^m}{2\Delta x} \\ \dfrac{\partial^2 u}{\partial x^2}\Big|_{i=n-2} = \dfrac{0 - 2u_{n-2}^m + u_{n-3}^m}{(\Delta x)^2} \end{cases} \tag{2a}$$

$$\begin{cases} u\dfrac{\partial u}{\partial x}\Big|_{i=1} = u_1^m \dfrac{-3u_1^m + 4u_2^m - u_3^m}{2\Delta x} \\ \dfrac{\partial^2 u}{\partial x^2}\Big|_{i=1} = \dfrac{u_2^m - 2u_1^m + 0}{(\Delta x)^2} \\ u\dfrac{\partial u}{\partial x}\Big|_{i=n-2} = u_{n-2}^m \dfrac{3u_{n-2}^m - 4u_{n-3}^m + u_{n-4}^m}{2\Delta x} \\ \dfrac{\partial^2 u}{\partial x^2}\Big|_{i=n-2} = \dfrac{0 - 2u_{n-2}^m + u_{n-3}^m}{(\Delta x)^2} \end{cases} \tag{2b}$$

Surprisingly, limit cycles can be seen in the case of $n = 102$. It is obvious that the finite difference scheme in which the fixed boundary values are included only in the viscous term causes the more complicated spurious numerical solutions. Figure 3 shows bifurcation diagrams of a discrete dynamical system generated by the downwind scheme (3). All of the absolute values of initial data of inner points are 0.131067, and the sign of each value is indicated in Fig. 3. Spatial discretizations which destabilize the system (e.g., a downwind scheme which includes a negative dissipative term) make those structures still more complicated. Conversely, in the case of spatial discretization which stabilizes the system, Eq. (4), no spurious numerical solution results.

$$\begin{aligned} u\frac{\partial u}{\partial x} &= \frac{u_i^m + |u_i^m|}{2}\frac{u_{i+1}^m - u_i^m}{\Delta x} + \frac{u_i^m - |u_i^m|}{2}\frac{u_i^m - u_{i-1}^m}{\Delta x} \\ &= u_i^m \frac{u_{i+1}^m - u_{i-1}^m}{2\Delta x} + \left(\frac{|u_i^m|\Delta x}{2}\right)\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta x)^2} \end{aligned} \tag{3}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} &= \frac{u_i^m + |u_i^m|}{2}\frac{u_i^m - u_{i-1}^m}{\Delta x} + \frac{u_i^m - |u_i^m|}{2}\frac{u_{i+1}^m - u_i^m}{\Delta x} \\ &= u_i^m \frac{u_{i+1}^m - u_{i-1}^m}{2\Delta x} - \left(\frac{|u_i^m|\Delta x}{2}\right)\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta x)^2} \end{aligned} \tag{4}$$

Conclusion

As already described, attention must be paid to the selection of Δt, initial and boundary conditions and reasonable schemes in order to obtain physically reasonable numerical solutions. The appearance of spurious numerical solutions is irrelevant to the accuracy of the scheme. Some types of errors, such as rounding error, truncation error, and so on, produce spurious numerical solutions. However, a perfect criterion to distinguish the spurious numerical solutions from the true solutions is not yet available. Though a mathematical approach is effective in the cases of ghost equilibrium fixed point and two-periodic points,[8] it is still quite difficult to judge whether or not a steady-state numerical solution corresponds to the true solution in those cases for which there is a possibility that spurious numerical solutions, such as limit cycles, tori, and chaotic motions, may appear. The several methods used in Refs. 3–6 and shown in the present paper are expected to be indispensable to studies in the field of computational fluid dynamics.

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